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LETTER TO THE EDITOR

Search for multifractality in damage spreading for Kauffman cellular automata

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Abstract. We check how an initial disturbance, called the damage, spreads through a square lattice of Kauffman cellular automata at their critical point. We determine the moments of the probability that a site has been damaged n times, and check for multifractality in the fractal dimensions, or critical exponents, of these moments of the damage probability.

The distribution of voltage drops in random resistor networks [1], or the distribution of growth probabilities in cluster growth processes [2], are two examples of multifractals ([3] and references therein). The kth moments of these voltages or probabilities have critical exponents or fractal dimensions which form an infinite set without any simple relation between these exponents. The present letter looks for analogous effects in the cluster growth probabilities of cellular automata. We selected the Kauffman model [4] for this study since the usual fractal dimensions [5] have already been studied for this model. The present letter deals with moments of the 'damage' spreading probabilities and parallels work [6] on moments of local periods in Kauffman models.

In the Kauffman model, as in all deterministic cellular automata, each 'spin' on a lattice is determined at time $t+1$ by the spin values of its neighbour sites at time t. The rule by which the neighbours govern the flipping of the central spin are selected randomly for each site from the total set of all possible functions. With probability p one selects, for each site separately, a rule giving the result 'up' for that neighbour configuration, and with probability $1-p$ one selects a 'down' rule. We restrict ourselves here to a nearest-neighbour spin-$\frac{1}{2}$ model on a square lattice. For further reviews of Kauffman models see [4, 5].

'Damage spreading' is a dynamic generalisation of the 'overlap' or Hamming distance concept [4, 5]. One looks at two identical lattices, with the same rules and initially the same spin configuration. After some stationary state has been established, one flips the centre spin. Then one continues to simulate the two lattices and observes, at every time step, which sites in the lattice have spins which differ in the two lattices. These sites are called 'damaged'. One calculates how often a site is damaged during the time development and how long it takes for the damage to reach the boundaries

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of the system. Analogous calculations have been made for other systems, such as Ising models [7].

We calculate here the probability $p_i = n_i/\Sigma_i n_i$ of a site to be damaged, where n_i is the total number of times this particular site i was damaged during the simulation. We average only over those samples where the damage touched the upper boundary of our square lattice, and found the minimum of all these times needed to reach the boundary. We ignore in our averages all times beyond this minimum time in order to reduce finite-size effects. We calculate, as a function of time, the moments $M_q = \Sigma_i p_i^q$ for various (not necessarily integral) q; of course, $M_q = 1$ for $q = 1$. For times much larger than unity we expect

$$M_q \propto t^{\phi(q)}. \tag{1}$$

For p far above the critical point p_c damage spreads easily over most of the system with a constant propagation velocity. Most lattice sites thus will have $n_i \propto t$ for sufficiently long times.

Since the number of damaged sites M_0 scales as $M_0 \propto t^{\tilde{d}_t}$ with d_t depending on the Euclidean dimension d [4, 5], we have $\Sigma_i n_i \propto t^{\tilde{d}_t+1}$. Thus the moments are expected to scale as $M_q = \Sigma_i (n_i/\Sigma n_i)^q \propto t^{\tilde{d}_t(1-q)}$ and

$$\phi(q) = \tilde{d}_t(1-q) \tag{2}$$

with $\tilde{d}_t = 2$ for $d = 2$.

Figure 1(a) shows selected moments $M_q(t)$ for $p = 0.5$ in lattices up to 100×100, averaged over 100 configurations. The asymptotic regime seems to be reached only after $t = 16$. This was checked by analysing systems with size up to 400. The slopes in this log-log plot for time t larger than 16, i.e. the effective exponents $\phi(q)$, are also indicated. In figure 2(a) we plot $\phi(q)$ against q. The data follow a straight line with slope $\tilde{d}_t \sim 2.1 \pm 0.2$ as expected from equation (2).

More interesting is the critical behaviour at $p = p_c$ where the damaged cluster just starts to spread over the whole lattice. The moments averaged over 1300 configurations with the relative slopes are shown in figure 1(b) while $\phi(q)$ against q is plotted in figure 2(b).

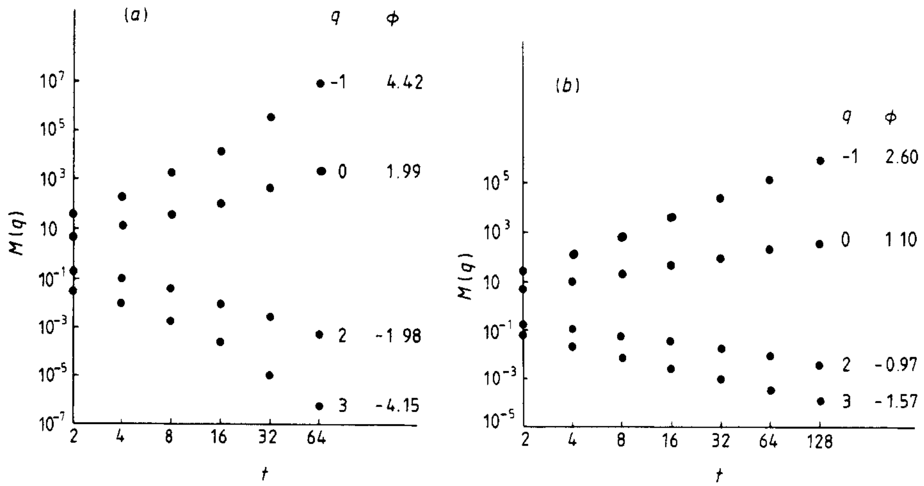

Figure 1. Log-log plot of moments M_q against time t, indicating the exponents $\phi(q)$ derived from their slopes. (a) $p = 0.5$, (b) $p = p_c = 0.29$.

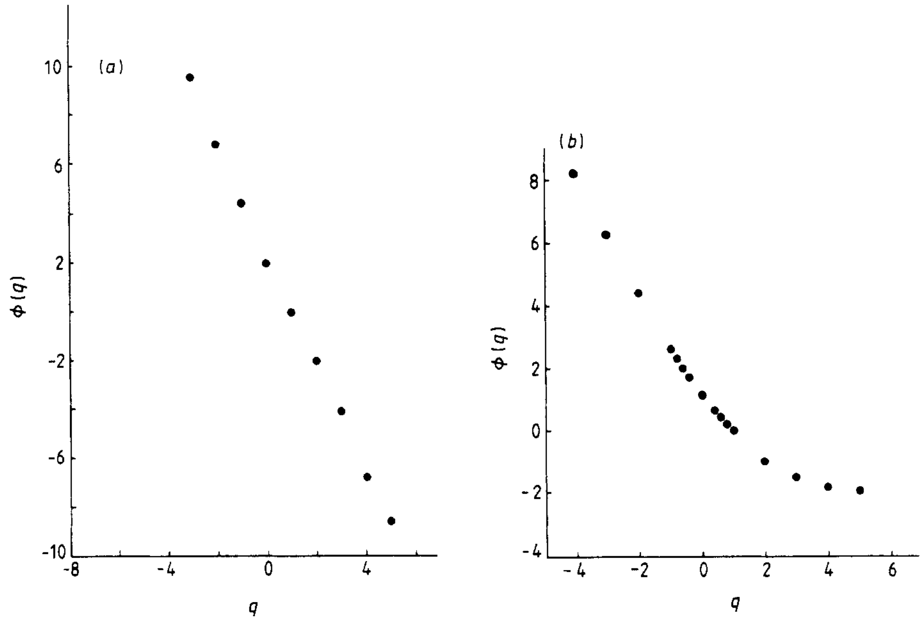

Figure 2. Critical exponents $\phi(q)$ against q. (a) $p=0.5$, (b) $p=p_c=0.29$.

If simple scaling would be valid the exponents $\phi(q)$ would be linear in q with possibly two straight lines, one dominated by the large values of p and the other by small values of p. Any departure from such behaviour is symptomatic of multifractality, namely the set of $\phi(q)$ is an infinite set of independent exponents, without any obvious relation between them [1-3].

The presence of this set of exponents implies that the fractal set of all the damage sites can be decomposed into infinitely many fractal subsets each made of sites characterised by the same probability p.

Our numerical data in figure 2(b) do not agree well with simple two-exponent scaling and seem to be consistent with the multifractality concept. The sample size employed here could be regarded as sufficient in a search for multifractality in random resistor networks [1] but might be relatively small for Kauffman models, where the total amount of damage up to the touching event is known [5] to follow no asymptotic power law for systems with less than 50×50 spins. In particular it was found [5] that the mass of the total damaged sites scale with an exponent $\tilde{d}_t=0.88$ for system sizes up to 200×200. This exponent should agree with the value $\phi(0)$. Here we find $\phi(0)\sim 1.1$ suggesting that we are not yet in the asymptotic regime. However, simulations for much bigger lattices would be very costly since already the present study took several hundred hours on a SUN 3/50 workstation.

Thus, for the lattice sizes employed here up to 100×100, multifractality seems to hold for the damage probabilities in the Kauffman model on the critical point of the square lattice. However one cannot completely exclude the possiblity that for larger systems a simple two-exponent scaling applies.

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